Introduction to Gradient-Based Direct Policy Search

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Outline

Notations

In this course, we use the classic reinforcement learning notations:

- $s \in \mathcal{S}$ for the states,
- $a \in \mathcal{A}$ for the actions,
- $V(s)$ for the state value function,
- $Q(s, a)$ for the state-action value function,
- $\pi(a|s)$ for the stationary stochastic policy,
- $\mu(s)$ for the stationary deterministic policy,
- $\arg\max$ gives a subset or a single value depending on the context.

Direct Policy Search

Markov decision process

An MDP is represented by its model $\mathcal{M} = (\mathcal{S}, \mathcal{A}, T, R, p_0, \gamma)$:

- States $s_t \in \mathcal{S}$,
- Actions $a_t \in \mathcal{A}$,
- Transition distribution $T(s_{t+1}|s_t, a_t)$,
- Reward function $r_t = R(s_t, a_t)$,
- Initial distribution $p_0(s_0)$,
- Discount factor $\gamma \in [0, 1[$.

In MDPs, states satisfy the Markov property:

$$\begin{aligned} p(s_{t+1}|s_0, a_0, \ldots, s_t, a_t) &= p(s_{t+1}|s_t, a_t) \\ &= T(s_{t+1}|s_t, a_t). \end{aligned}$$

Stochastic policies in MDPs

Definition (Stationary stochastic policy)

A stationary stochastic policy $\pi \in \Pi = \mathcal{S} \to \Delta(\mathcal{A})$ is a mapping from a state to a distribution over the actions, whose density writes $\pi(a_t|s_t)$.

$$V^{\pi}(s) = \mathop{\mathbb{E}}_{\substack{a_t \sim \pi(\cdot|s_t) \\ s_{t+1} \sim T(\cdot|s_t, a_t)}} \left[\sum_{t=0}^{\infty} \gamma^t R(s_t, a_t) \middle| s_0 = s \right]$$

Theorem (Optimality of stationary stochastic policies in MDPs)

There exists an optimal stationary stochastic policy.

Direct Policy Search

Do not solve a more general problem as an intermediate step.

— Vladimir Vapnik, 1998

As we care about optimal behaviour, why not directly learning a policy ?

Definition (Problem Statement)

In Direct Policy Search we look for a policy $\pi^* \in \Pi$ maximizing the expect discounted sum of rewards (i.e., the expected return of the policy):

$$J(\pi) = \mathop{\mathbb{E}}_{\substack{s_0 \sim p_0(\cdot) \\ a_t \sim \pi(\cdot|s_t) \\ s_{t+1} \sim T(\cdot|s_t, a_t)}} \left[\sum_{t=0}^{\infty} \gamma^t R(s_t, a_t) \right].$$

Questions :

1. Is this policy optimal in the sense of Bellman ?
2. Is a Bellman optimal policy optimal for DPS ?

Direct Policy Search – Advantages

Policy-based RL has several advantages compared to value-based RL:

1. We optimize the true control objective.
2. It extends to continuous state-action spaces.
3. Sometimes simple behaviours are optimal while value functions are complex.

We will focus on stochastic policies as their expected return is usually smoother than deterministic policies.

Policy-Gradient Methods

Policy Gradient Methods – Recipe

Direct policy search is usually solved with policy-gradient methods.

1. We represent the policy with a differentiable parametric function π_θ.
2. We perform stochastic gradient ascent on the expected return.

Policy Gradient Methods – Policy Parameterization

We commonly use Gaussian policies in which actions are draw as

$$a_t \sim \mathcal{N}(\cdot|\mu_\theta(s_t), \Sigma_\theta(s_t)).$$

How to represent such a distribution with a neural network ?

Policy Gradient Theorem

$$\nabla_\theta J(\pi_\theta) = \nabla_\theta \mathop{\mathbb{E}}_{\substack{s_0 \sim p_0(\cdot) \\ a_t \sim \pi_\theta(\cdot|s_t) \\ s_{t+1} \sim T(\cdot|s_t, a_t)}} \left[\sum_{t=0}^{\infty} \gamma^t R(s_t, a_t) \right]$$

1. How to compute the gradient ?
2. Gradient of a Monte-Carlo estimates of J will not work... Why?

Policy Gradient Theorem

Theorem (Policy Gradient Theorem)

For any differentiable policy π_θ, the policy gradient of $J(\pi_\theta)$ is [Sutton et al., 1999]

$$\nabla_\theta J(\pi_\theta) = \mathop{\mathbb{E}}_{\substack{s_0 \sim p_0(\cdot) \\ a_t \sim \pi_\theta(\cdot|s_t) \\ s_{t+1} \sim T(\cdot|s_t, a_t)}} \left[\sum_{t=0}^{\infty} \gamma^t Q^{\pi_\theta}(s_t, a_t) \nabla_\theta \log \pi_\theta(a_t|s_t) \right],$$

where

$$Q^{\pi_\theta}(s, a) = \mathop{\mathbb{E}}_{\substack{a_t \sim \pi_\theta(\cdot|s_t) \\ s_{t+1} \sim T(\cdot|s_t, a_t)}} \left[\sum_{t=0}^{\infty} \gamma^t R(s_t, a_t) \middle| s_0 = s, a_0 = a \right].$$

- How to approximate the state-action value function Q^{π_θ} ?
- How to approximate the expectation ?

Likelihood Ratio PG – REINFORCE

Using Monte-Carlo over n i.i.d. trajectories, we get

$$\hat{\nabla}_\theta J(\pi_\theta) = \left\langle \sum_{t=0}^{T-1} \gamma^t \nabla_\theta \log \pi_\theta(a_t|s_t) \hat{Q}_t \right\rangle_n$$

$$\hat{Q}_t = \sum_{t'=t}^{T-1} \gamma^{t'-t} r_t \ .$$

- Is this estimate unbiased when $T \to \infty$, why ?
- What is the influence of the horizon T on the gradient ?
- In practice we neglect the γ^t in the gradient expression...
- Show that $\|Q^{\pi_\theta} - \mathbb{E}[\hat{Q}_t]\| \leq \frac{\gamma^{T-t}}{1-\gamma} \max_{s,a} R(s,a)$.

Proof Property

REINFORCE algorithm

In summary, the REINFORCE algorithm writes as follows.

Algorithm 1: REINFORCE algorithm

Initialise θ randomly.
for $k \leftarrow 1, \ldots, K$ **do**
    Sample n trajectories with the current policy in the MDP
    Update $\theta_k = \theta_{k-1} + \alpha_k \hat{\nabla}_\theta J(\pi_\theta)$

Variance Reduction and Actor-Critic Methods

Baseline

- The gradient estimate can be subject to a large variance !
- Subtracting a baseline from the cumulative reward can decrease the variance.

$$\hat{\nabla}_\theta J(\pi_\theta) = \left\langle \sum_{t=0}^{T-1} \gamma^t \left(Q_t - b_t\right) \nabla_\theta \log \pi_\theta(a_t|s_t) \right\rangle_n$$

- In practice, it is common to choose the mean cumulative reward

$$b_t = \left\langle \sum_{t'=t}^{T-1} \gamma^{t'-t} r_t \right\rangle_n .$$

Baseline in General

Baselines keep the gradient estimate unbiased !

Theorem (Policy Gradient Theorem with Baseline)

For any differentiable policy π_θ, for any function of the state f, the policy gradient of $J(\pi_\theta)$ is

$$\nabla_\theta J(\pi_\theta) = \mathop{\mathbb{E}}_{\substack{s_0 \sim p_0(\cdot) \\ a_t \sim \pi_\theta(\cdot|s_t) \\ s_{t+1} \sim T(\cdot|s_t, a_t)}} \left[\sum_{t=0}^{\infty} \gamma^t (Q^{\pi_\theta}(s_t, a_t) - f(s_t)) \nabla_\theta \log \pi_\theta(a_t|s_t) \right].$$

When we use the mean cumulative rewards as baseline, we use an approximation of the state value function !

Proof Theorem

Advantage Actor Critic

- Actor-Critic Algorithms use a function approximator (the critic) representing $Q^{\pi_\theta}(s_t, a_t) - f(s_t)$!
- Advantage Actor-Critic (A2C) learns the value function V_ϕ of the current policy [Mnih et al., 2016]

$$\hat{\nabla}_\theta J(\pi_\theta) = \left\langle \sum_{t=0}^{T-1} \gamma^t \left((\sum_{t'=t}^{T-1} \gamma^{t'-t} r_{t'} + \gamma^T V_\phi(s_T)) - V_\phi(s_t) \right) \nabla_\theta \log \pi_\theta(a_t|s_t) \right\rangle_n .$$

How to learn the parameters of V_ϕ ?

Value Function Evaluation with Monte-Carlo

The first approach is Monte-Carlo Learning !

$$\min_{\phi} \left(V_{\phi}(s) - \mathop{\mathbb{E}}_{\substack{a_t \sim \pi_{\theta}(\cdot|s_t) \\ s_{t+1} \sim T(\cdot|s_t, a_t)}} \left[\sum_{t=0}^{\infty} \gamma^t R(s_t, a_t) \middle| s_0 = s \right] \right)^2 \forall s$$

In practice we perform gradient descent on the empirical estimate for each state encountered:

$$\mathcal{L}(\phi) = \left\langle \sum_{t=0}^{T-1} \left(V_{\phi}(s_t) - \sum_{t'=t}^{T-1} \gamma^{t-t'} r_{t'} \right)^2 \right\rangle_n .$$

This approach is unbiased but subject to high variance !

Value Function Evaluation with TD-Learning

The second approach is Temporal-Difference (TD) Learning !

$$\min_{\phi} \left(V_\phi(s) - \mathop{\mathbb{E}}_{\substack{a_t \sim \pi_\theta(\cdot|s_t) \\ s_{t+1} \sim T(\cdot|s_t, a_t)}} \left[R(s_t, a_t) + \gamma V_\phi(s_{t+1}) | s_t = s \right] \right)^2 \forall s$$

In practice we perform (quasi) gradient descent on the empirical estimate for each state encountered:

$$\mathcal{L}(\phi) = \left\langle \sum_{t=0}^{T-1} \left(V_\phi(s_t) - r_t - \gamma V_\phi(s_{t+1}) \right)^2 \right\rangle_n .$$

This approach is more stable but provides biased value functions !

Value Function Evaluation with Multi-step TD learning

In practice we combine both worlds with multi-step TD-learning and minimize the following loss:

$$\mathcal{L}(\phi) = \left\langle \sum_{t=0}^{T-1} \left(V_\phi(s_t) - \sum_{t'=t}^{T-1} \gamma^{t-t'} r_{t'} - \gamma^{T-t} V_\phi(s_T) \right)^2 \right\rangle_n .$$

The update direction is

$$\hat{\nabla}\mathcal{L}(\phi) = \left\langle \left(\sum_{t=0}^{T-1} V_\phi(s_t) - \sum_{t'=t}^{T-1} \gamma^{t-t'} r_{t'} - \gamma^{T-t} V_\phi(s_T) \right) \left(\sum_{t=0}^{T-1} \nabla_\phi V_\phi(s_t) \right) \right\rangle_n .$$

A2C algorithm

In summary, the A2C algorithm writes as follows.

Algorithm 2: A2C algorithm

Initialise θ randomly.
for $k \leftarrow 1, \ldots, K$ **do**
    Sample n trajectories with the current policy in the MDP
    Update $\phi_k = \phi_{k-1} - \alpha_k \hat{\nabla}_\theta \mathcal{L}(\phi)$
    Update $\theta_k = \theta_{k-1} + \beta_k \hat{\nabla}_\theta J(\pi_\theta)$

A2C algorithm

- This algorithm is more sample efficient... Why ?
- It is said to be an on-policy algorithm.
- As such, the algorithm is prone to converge towards local extrema...

Entropy Regularization

Local Optimality

- Large variance decreases the expected return of the policy.
- In practice the gradient ascent thus tends to reduce the variance.
- The policy converges towards a deterministic policy.
- The policy has a larger but less concave return...

The gradient ascent converges to a locally optimal deterministic policy !

Avoiding Local Optimality – Variance Control

A simple approach is to add a constant disturbance to the actions, a Gaussian policy would provide actions distributed as

$$a_t \sim \mathcal{N}(\cdot|\mu_\theta(s_t), \Sigma_\theta(s_t) + \Lambda_k).$$

How to increase the variance of other distributions (e.g., mixture, beta-distribution, normalizing flow) ?

Avoiding Local Optimality – Entropy Regularization

The preferred approach is to provide an entropy bonus $\mathcal{H}(\pi_\theta)$ to the return.

Algorithm 3: A2C algorithm with entropy regularization

Initialise θ randomly.
for $k \leftarrow 1, \dots, K$ **do**
    Sample n trajectories with the current policy in the MDP
    Update $\phi_k = \phi_{k-1} - \alpha_k \hat{\nabla}_\theta \mathcal{L}(\phi)$
    Update $\theta_k = \theta_{k-1} + \beta_k \hat{\nabla}_\theta J(\pi_\theta) + \lambda_k \nabla_\theta \mathcal{H}(\pi_\theta)$

Conclusions

Conclusion

In summary:

- We introduced direct policy search.
- We saw how to optimize a policy with the PG Theorem.
- Variance reduction lead us to the A2C algorithm.
- Entropy regularization enhances the performance of A2C.

Next week:

- We will dive into more complex on-policy algorithms.
- We will see a first off-policy method.

References

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